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Bounds on the Density of States of Random Schrödinger Operators

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Bounds are obtained on the unintegrated density of states $\rho(E)$ of random Schrödinger operators $H = -\Delta + V$ acting on $L^2(\mathbb{R}^d)$ or $l^2(\mathbb{Z}^d)$. In both cases the random potential is

$$V := \sum_{y \in \mathbb{Z}^d} V_y \chi(\Lambda(y))$$

in which the $\{V_y\}_{y \in \mathbb{Z}^d}$ are IID random variables with density f. The χ denotes indicator function, and in the continuum case the $\{\Lambda(y)\}_{y \in \mathbb{Z}^d}$ are cells of unit dimensions centered on $y \in \mathbb{Z}^d$. In the finite-difference case $\Lambda(y)$ denotes the site $y \in \mathbb{Z}^d$ itself. Under the assumption $f \in L_0^{1+\varepsilon}(\mathbb{R})$ it is proven that in the finite-difference case $\rho \in L^\infty(\mathbb{R})$, and that in the $d = 1$ continuum case $\rho \in L^\infty_{\text{loc}}(\mathbb{R})$.

KEY WORDS: Random operators; density of states; Schrödinger equation.

1. INTRODUCTION

In the last few years, random Schrödinger operators have received rigorous mathematical treatment. Such operators are of the form $H = -\Delta + V$ and act on either $l^2(\mathbb{Z}^d)$ or $L^2(\mathbb{Z}^d)$, with Δ being, respectively, the finite-difference or the continuum Laplacian. The potential is defined to be

$$V(\cdot) := \sum_{x \in \mathbb{Z}^d} V_x \chi_{\Lambda(x)}(\cdot)$$

in which $\chi_{\Lambda(x)}$ denotes the indicator function of the set $\Lambda(x)$. In the finite-difference case $\Lambda(x)$ means the site $x \in \mathbb{Z}^d$; in the continuum case, the hybercubic cell of unit dimensions centered at x. Henceforth the $\{V_x\}_{x \in \mathbb{Z}^d}$ will be taken to be independent, identically distributed random variables,

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each with distribution $f(V)\,dV$. I assume that the density f exists as a function.

Results on the finite-difference case include results on the falloff of the Green's function $(z-H)^{-1}(x, y)$ as $|x-y| \to \infty$,[2] and on the boundedness of the unintegrated density of states (DOS) of H.[5] In this paper I report on a considerable extension of the previously obtained bounds on the DOS. The new bounds are obtained by a novel technique involving integration over the isospectral varieties of spatially cutoff versions of H, introduced in Section 2. They are stated in Theorem 1 of Section 3 and Theorem 2 of Section 4. The first deals with the finite-difference case; the latter, with the $d=1$ continuum case. In both cases it is proven that the DOS is bounded if $f \in L_0^{1+\varepsilon}(\mathbb{R})$.

2. THE DENSITY OF STATES AND ITS INTEGRAL REPRESENTATION

To begin with, recall how the DOS is defined. The following definitions apply to both the finite-difference and continuum cases.

If $\Lambda \subset \mathbb{Z}^d$, define $\bar{\Lambda}$ by

$$\bar{\Lambda} := \begin{cases} \Lambda & \text{finite-difference case} \\ \bigcup_{y \in \Lambda} \Lambda(y) & \text{continuum case} \end{cases}$$

and let H_Λ be the finite-volume restriction of H to $\bar{\Lambda}$. In the finite-difference case, $H_\Lambda := \chi_{\bar{\Lambda}} H \chi_{\bar{\Lambda}}$; in the continuum case one can take $H_\Lambda := -\Delta_{\bar{\Lambda},D} + V$, with the Laplacian given Dirichlet boundary conditions on $\partial\bar{\Lambda}$. For every $\sigma \in \Sigma$, the class of Borel subsets of $\mathbb{R}$, define $P(H; \sigma)$ and $P(H_\Lambda; \sigma)$ to be the spectral projections of H and H_Λ on σ. The *DOS measure associated to the volume* Λ is then defined to be the random measure

$$N_\Lambda(\cdot) := |\Lambda|^{-1} \operatorname{tr} P(H_\Lambda; \cdot)$$

This measure is purely atomic, with an atom at every eigenvalue of H_Λ. In the finite-difference case $N_\Lambda(\cdot)$ has total mass unity.

It is a consequence of the ergodicity of translations on the underlying probability space[3] that if $\sigma \in \Sigma$ is sufficiently nice,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} N_\Lambda(\sigma)$$

exists almost surely and in expectation, and is independent of the choice of $\Lambda \uparrow \mathbb{Z}^d$. Moreover, this limit equals

$$\mathscr{E} \operatorname{tr}[\chi_{\Lambda(0)} P(H; \sigma)\, \chi_{\Lambda(0)}]$$

if $\mathscr{E}$ denotes expectation. The *infinite-volume DOS measure* $N(\cdot)$ is accordingly defined to be the nonrandom Borel measure

$$\mathscr{E}\ \mathrm{tr}[\chi_{\Lambda(0)} P(H;\cdot)\,\chi_{\Lambda(0)}]$$

It contains no pure point component,[1] and by ergodicity[6] its support lies within $\mathrm{supp}(-\Delta)+\mathrm{supp}\,f$. It will be written as $\rho(E)\,dE$. We call $\rho(\cdot)$ the *infinite-volume DOS*, or simply the DOS. It exists *a priori* only as a distribution, but under weak hypotheses may be shown to be a function as follows.

The averaged finite-volume DOS measures $\mathscr{E}N_\Lambda(\cdot)$ are absolutely continuous if f is, and may be written as $\mathscr{E}\rho_\Lambda(E)\,dE$. The $\mathscr{E}\rho_\Lambda(\cdot)$ may be termed the *average finite-volume DOSs*; they necessarily exist as functions. If $\|\mathscr{E}\rho_\Lambda\|_\infty$ is bounded uniformly in Λ, then by the $\Lambda\uparrow\mathbb{Z}^d$ convergence of $\mathscr{E}N_\Lambda$ to $\rho(E)\,dE$, ρ will exist as a function and obey the same L^∞ bound. The fact that $\|\mathscr{E}\rho_\Lambda\|_\infty$ is under weak hypotheses bounded uniformly in Λ is a new result of this paper, stated in Theorems 1′ and 2′.

I shall prove these theorems with the aid of an easily understood, but little known integral representation for $\mathscr{E}\rho_\Lambda(\cdot)$. If $E\in\mathbb{R}$, define a random surface $\mathscr{S}_E^\Lambda\subset\mathbb{R}^\Lambda$ by

$$\mathscr{S}_E^\Lambda := \{\underline{V}\in\mathbb{R}^\Lambda \mid E\in\sigma(H_\Lambda)\} \tag{1}$$

In the finite-difference case, $\mathscr{S}_E^\Lambda$ is an algebraic variety, but in the continuum case it is analytic rather than algebraic. It could be termed an "isoeigenvalue surface," for it is the closest thing to an isospectral manifold that can be defined in the space $\mathbb{R}^\Lambda$ of potentials on Λ. At all points on it, $H_\Lambda = H_\Lambda(\underline{V})$ has an eigenvalue at E. Degenerate eigenvalues give rise to self-intersections, so $\mathscr{S}_E^\Lambda$ is not in general an embedded submanifold of $\mathbb{R}^\Lambda$.

Wegner[5] observed that $\mathscr{E}\rho_\Lambda(E)$ can be expressed as an integral over $\mathscr{S}_E^\Lambda$. To see this, regard the *integrated* DOS $N_\Lambda([E,E+\delta))$ as a random variable on the probability space $(\mathbb{R}^\Lambda, \prod_{x\in\Lambda} f(V_x)\,dV_x)$. Its expectation $\mathscr{E}N_\Lambda([E,E+\delta])$ equals

$$|\Lambda|^{-1}\int_{\underline{V}\in\mathbb{R}^\Lambda} n_\Lambda([E,E+\delta))(\{\underline{V}\}_{x\in\Lambda}) \prod_{x\in\Lambda} f(V_x) \tag{1.5}$$

in which $n_\Lambda([E,E+\delta))$ is the number of eigenvalues of $-\Delta_{\bar{\Lambda},D}+\underline{V}$ lying in $[E,E+\delta)$, namely the number of times the line segment extending from $\underline{V}$ to $\underline{V}+\delta(1,\ldots,1)$ in $\mathbb{R}^\Lambda$ intersects $\mathscr{S}_E^\Lambda$. The differential version of (1.5) therefore expresses $\mathscr{E}\rho_\Lambda(E)$, the density of $\mathscr{E}N_\Lambda(\cdot)$, as an integral over $\mathscr{S}_E^\Lambda$:

$$\mathscr{E}\rho_\Lambda(E) = |\Lambda|^{-1}\int_{\underline{V}\in\mathscr{S}_\Lambda^E} |(1,\ldots,1)\cdot\underline{N}(\underline{V})| \prod_{x\in\Lambda} f(V_x)\,dA$$

Here dA is surface area on $\mathscr{S}_E^A$, and $\underline{N}(\underline{V})$ is the unit normal to $\mathscr{S}_E^A$ at $\underline{V}$. We define $\underline{e} := |A|^{-1}(1,..., 1)$ so that

$$\mathscr{E}\rho_A(E) = \int_{\underline{V} \in \mathscr{S}_A^E} \left| \sum_{x \in A} e_x N_x \right| \prod_{x \in A} f(V_x)\, dA = \int_{\underline{V} \in \mathscr{S}_A^E} \prod_{x \in A} f(V_x)\, d\mu_A(\underline{V}) \qquad (2)$$

in which the surface measure $d\mu_A(\underline{V}) := |\underline{e} \cdot \underline{N}(\underline{V})|\, dA$.

Note that by first-order Rayleigh–Schrödinger perturbation theory, if $\psi(\cdot)$ is the eigenfunction of $H_A = H_A(\underline{V})$ with eigenvalue E, then at $\underline{V} \in \mathscr{S}_E^A$ the components N_y of the unit normal $\underline{N}$ to $\mathscr{S}_E^A$ are proportional to

$$\begin{cases} |\psi(y)|^2 & \text{finite-difference case} \\ \int_{A(y)} |\psi(x)|^2\, dx & \text{continuum case} \end{cases} \qquad (2.5)$$

As these components are nonnegative, we shall in the sequel drop the absolute value signs from the definition of the surface measure $d\mu_A(\underline{V})$, and write $d\mu_A(\underline{V}) = \underline{e} \cdot \underline{N}(\underline{V})\, dA$.

3. RESULTS IN THE FINITE-DIFFERENCE CASE

For pedagogical purposes, consider the simple case of $A = \{x_1, x_2\}$, for x_1, x_2 adjacent sites in $\mathbb{Z}^d$. Our finite-difference Laplacian Δ is defined to have matrix elements

$$\Delta(x, y) = \begin{cases} 1 & |x - y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and has spectrum $\sigma(\Delta) = [-2d, 2d]$. So, by definition, $\mathscr{S}_E^{\{x_1, x_2\}} \subset \mathbb{R}^2$ is the set of (V_{x_1}, V_{x_2}) for which the matrix $H_{\{x_1, x_2\}}$, i.e.,

$$\begin{pmatrix} V_{x_1} & -1 \\ -1 & V_{x_2} \end{pmatrix}$$

has eigenvalue E. This subset of $\mathbb{R}^2$ is a hyperbola. (Indeed, so is the intersection of any isoeigenvalue surface $\mathscr{S}_E^A$ with a two-dimensional hyperplane parallel to two of the coordinate axes in $\mathbb{R}^A$.) Also,

$$\mathscr{S}_E^A = E(1, 1) + \mathscr{S}_0^A$$

So the representation (2) implies

$$\mathscr{E}\rho_{\{x_1, x_2\}}(E) = \int_{\mathscr{S}_0^{\{x_1, x_2\}} \subset \mathbb{R}^2} f(V_{x_1} - E)\, f(V_{x_2} - E)\, d\mu(\underline{V}) \qquad (3)$$

Proposition 1. Let Λ comprise two adjacent sites, and suppose $f \in L^2(\mathbb{R})$ and $\operatorname{supp} f \subset [-C, C]$. Then for some constant $c(C)$, $\mathscr{E}\rho_\Lambda(E) \leqslant c(C) \|f\|_2^2$ for all E.

Proof. On any compact subset of a hyperbola, the two components N_{x_1} and N_{x_2} of the unit normal vector are bounded away from zero. In particular, on $\mathscr{S}_0^{\{x_1, x_2\}} \cap [-C, C]^2$, N_{x_1} and N_{x_2} are bounded below by some $\gamma(C) > 0$.

So, if one writes (3) as an integral with respect to V_{x_1}, i.e.,

$$\int f(V_{x_1} - E)\, f(V_{x_2}(V_{x_1}) - E)\, J(V_{x_1})\, dV_{x_1}$$

the Jacobian factor $J(V_{x_1}) = 1 + N_{x_1} N_{x_2}^{-1}$ [i.e., $d\mu(\underline{V})/dV_{x_1}$] will be bounded by $1 + \gamma^{-1}$ on the interval $-C \leqslant V_{x_1} \leqslant C$. Moreover, on this interval the L^2 norm of $f(V_{x_2}(\cdot) - E)$ will be bounded by

$$\sup_{-C \leqslant V_1 \leqslant C} (N_{x_1}^{-1} N_{x_2}) \|f\|_2 \leqslant \gamma^{-1} \|f\|_2$$

So the proposition holds by the Schwartz inequality, with $c = \gamma^{-1} + \gamma^{-2}$. ∎

What is the meaning of Proposition 1? Equation (3) expresses $\mathscr{E}\rho_{\{x, x_2\}}(E)$ as a *twisted convolution* of f with itself. Proposition 1 is therefore analogous to $\| f * f \|_\infty \leqslant \| f \|_2^2$, which holds for the conventional convolution $*$.

In general, (2) expresses $\mathscr{E}\rho_\Lambda(E)$ as a $|\Lambda|$-fold twisted convolution of f with itself. But a special case of Young's inequality[4] states that $\| f^{*p} \|_\infty \leqslant \| f \|_{p'}^p$, in which $p' := (1 - p^{-1})^{-1}$. So one would expect that for arbitrarily small $\varepsilon > 0$, if Λ is sufficiently large, then $\|\mathscr{E}\rho_\Lambda(\cdot)\|_\infty$ could be bounded in terms of $\| f \|_{1+\varepsilon}$. In other words, increasing the size of Λ should eventually *smooth* $\mathscr{E}\rho_\Lambda$ into an L^∞ function, even if f is only $L^{1+\varepsilon}$. This smoothing is the subject of the following theorem, the finite-difference half of the principal result of this paper.

Theorem 1. Suppose $\operatorname{supp} f \subset [-C, C]$. In the finite-difference case, for all integral $p \geqslant 1$, there exists a constant $c_p = c_p(C)$ such that $\|\rho\|_\infty \leqslant c_p \| f \|_{p'}^p$, $p' := (1 - p^{-1})^{-1}$.

Remark. This theorem generalizes the above-mentioned result of Wegner,[5] who proved $\|\rho\|_\infty \leqslant \| f \|_\infty$ (i.e., the $p = 1$ case of the theorem) in the finite-difference case. Note that by the above-mentioned result of Kunz and Souillard,[6] $\operatorname{supp} \rho \subset [-C - 2d, C + 2d]$.

Proof. As stated, it suffices to show that the bounds on ρ hold with ρ replaced by $\mathscr{E}\rho_\Lambda$, for Λ arbitrarily large, i.e., that the following theorem holds.

Theorem 1′. Suppose $\operatorname{supp} f \subset [-C, C]$. For all integral $p \geqslant 1$ one can choose a sequence $\{\Lambda(N; p)\}_{N=1}^{\infty}$ of boxes converging to $\mathbb{Z}^d$ such that for some $c_p = c_p(C)$,

$$\|\mathscr{E}\rho_{\Lambda(N;p)}\|_{\infty,[-C-2d,C+2d]} \leqslant c_p \|f\|_{p'}^{p} \qquad \text{for all} \quad N$$

Remark. For ease of understanding I shall prove the $p=2$ case first. The proof resembles the above $|\Lambda| = 2$ treatment.

Proof of Theorem 1′, $p=2$ case. The proof will write the integral over $\mathscr{S}_E^\Lambda$ for $\mathscr{E}\rho_\Lambda$ in terms of the integrals over the plane curves that are the intersections of $\mathscr{S}_E^\Lambda$ with two-dimensional hyperplanes parallel to the coordinate axes. The following lemmas will be used to bound these integrals.

Lemma 1. Let $\mathscr{S}$ be a differentiable curve in $\mathbb{R}^2$. Suppose that: (i) the unit normal to $\mathscr{S}$ has both components nonnegative and bounded below by $\beta > 0$. (ii) S intersects any line parallel to either coordinate axis at most k times.

Then there exists a constant $c = c(\beta)$ such that

$$\int_{\underline{V} \in \mathscr{S}} f(V_1)\, f(V_2)\, g(\underline{V})\, dl \leqslant kc\, \|f\|_2^2\, \|g\|_\infty \tag{4}$$

Remark. This lemma was implicitly used in the proof of Proposition 1. There $k=1$, as $\mathscr{S}$ was a hyperbola aligned with the coordinate axes.

Lemma 2. For some $\beta(C, d) > 0$ and some finite set $\Lambda_0' \subset \mathbb{Z}^d$ the following is true. If $\Lambda_0 \subset \Lambda$ is congruent to Λ_0' and ψ is an eigenfunction of H_Λ with eigenvalue $E \in [-C-2d, C+2d]$, then there exist sites x_1, $x_2 \in \Lambda_0$ such that $|\psi_{\Lambda_0}(x_k)| \geqslant \beta$, $k = 1, 2$. Here ψ_{Λ_0} denotes the normalized (in the l^2 sense) restriction of ψ to Λ_0.

Remark. By first-order Rayleigh–Schrödinger perturbation theory, the 2-vector $(|\psi_{\Lambda_0}(x_1)|^2, |\psi_{\Lambda_0}(x_2)|^2)$ is normal to the plane curve obtained by intersecting $\mathscr{S}_E^\Lambda$ with the two-dimensional hyperplane of fixed $\{V(x)\}_{x \in \Lambda, x \neq x_1, x_2}$. This justifies the detailed study of $|\psi|\,(\cdot)$.

Proof of Lemma 2. The Λ_0' of the lemma will be chosen to satisfy the following adjacency condition: for all $z \in \Lambda_0'$, there exists $y \in \Lambda_0' \backslash \partial\Lambda_0'$ such that $|y - z| = 1$. Here $\partial\Lambda_0'$ denotes the set of boundary sites of Λ_0', i.e., those distant by exactly one unit from a site in $\mathbb{Z}^d \backslash \Lambda_0'$.

Assume that the lemma were false, i.e., that for an eigenfunction ψ of H_Λ, ψ_{Λ_0} could be arbitrarily concentrated on some site $z \in \Lambda_0$. That is, $\psi_{\Lambda_0}(z)$ could be arbitrarily close to 1. By choice there exists $y \in \Lambda_0 \backslash \partial \Lambda_0$ such that $|y-z| = 1$. The eigenvalue equation then implies

$$\psi_{\Lambda_0}(y) = (V_y - E)^{-1} \sum_{|x-y|=1} \psi_{\Lambda_0}(x)$$

But of the 2^d terms in this summation, one will be $\psi_{\Lambda_0}(z)$. As $|V_y - E|^{-1} \geqslant (2C+2d)^{-1}$, on at least one of the 2^d elements of $\{x \mid x \neq z, |x-y| = 1\} \cup \{y\}$, $|\psi_{\Lambda_0'}(\cdot)|$ would take on a value greater than some $\beta(C, d) > 0$. This would contradict the assumption.

Our adjacency condition on Λ_0' may be satisfied by choosing

$$\Lambda_0' := \{\underline{x} \in \mathbb{Z}^d \mid 1 \leqslant x_j \leqslant 4 \text{ and } x_j = 1 \text{ or } 4 \text{ for at most one } j\}$$

For example, if $d = 1$, then Λ_0' could be an interval of four consecutive sites. ∎

Given the set Λ_0' of Lemma 2, choose $\Lambda(N; p)$ to be a box $\Lambda := \bigcup_{j=1}^N \Lambda_j$, i.e., a connected union of N translates of Λ_0'. The union need not be disjoint, but I shall require

$$|\Lambda| > \alpha_1 N |\Lambda_0'| \tag{5}$$

for some $\alpha_1 > 0$.

Definition. If $A \subset \Lambda$ and $\{V_x\}_{x \in \Lambda \setminus A}$ are specified, one defines $\mathscr{S}_E^{\Lambda,A}$ to be the intersection of $\mathscr{S}_E^\Lambda$ with the $|A|$-dimensional hyperplane in $\mathbb{R}^\Lambda$ of fixed $\{V_x\}_{x \in \Lambda \setminus A}$. One can regard $\mathscr{S}_E^{\Lambda,A}$ as a $\{V_x\}_{x \in \Lambda \setminus A}$-dependent surface in $\mathbb{R}^A$.

The particular form of the surface measure $d\mu(\underline{V})$ in (2), i.e., $\underline{\varrho} \cdot \underline{N}\, dA$, allows a powerful change of variables. If $N_x \neq 0$ for at least one $x \in A$, then the unit normal to $\mathscr{S}_E^{\Lambda,A}$ in $\mathbb{R}^A$ will be defined and equal

$$\{n_x\}_{x \in A} := \left(\sum_{x \in A} N_x^2\right)^{-1/2} \{N_x\}_{x \in A}$$

So if $A \subset \Lambda$, equip $\mathscr{S}_E^{\Lambda,A}$ with surface measure

$$d\mu^A(\underline{V}) = \begin{cases} \underline{\varrho}_A \cdot \underline{n}\, dA & \text{if } \quad N_x \neq 0 \text{ for some } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

in which $e_A = |A|^{-1}(1,..., 1) \in \mathbb{R}^A$. Then, if $A = \mathring{\bigcup}_{j=1}^{M} A^{(j)}$ (a *disjoint* union), Eq. (2) may be rewritten as

$$\mathscr{E}\rho_A(E) = \sum_{j=1}^{M} \frac{|A^{(j)}|}{|A|} \int \prod_{z \in A \setminus A^{(j)}} f(V_z)\, dV_z \int_{\underline{V} \in \mathscr{S}_E^{A,A^{(j)}}} \prod_{y \in A^{(j)}} f(V_y)\, d\mu^{(j)}(\underline{V}) \tag{6}$$

in which $d\mu^{(A_j)}$ has been written as $d\mu^{(j)}$. This is because the Jacobian factor

$$\frac{dA(\underline{V})}{dA^{(j)}(\underline{V}) \prod_{z \in A \setminus A^{(j)}} dV_z} = \left(\sum_{y \in A^{(j)}} N_y^2 \right)^{-1/2}$$

when multiplying the $\sum_{x \in A^{(j)}} e_x N_x$ terms in (2), gives

$$\frac{|A^{(j)}|}{|A|} \underline{e}_{A_j} \cdot \underline{n}$$

which is precisely the factor in (6). This is true for all j, and the summation over j exhausts the terms in (2).

As $A = \bigcup_{\alpha=1}^{N} A_j$ (a not necessarily disjoint union), the equality (6) is replaced by an inequality, due to double counting of the contributions from sites in more than one $A^{(j)}$. We have

$$\begin{aligned}
\mathscr{E}\rho_A(E) &\leqslant \frac{|A_0'|}{|A|} \sum_{j=1}^{N} \int \prod_{z \in A \setminus A_j} f(V_z)\, dV_z \int_{\underline{V} \in \mathscr{S}_E^{A,A_j}} \prod_{y \in A_j} f(V_y)\, d\mu^{(j)}(\underline{V}) \\
&\leqslant \frac{N\,|A_0'|}{|A|} \sup_{1 \leqslant j \leqslant N} \sup_{\underline{V} \in \mathbb{R}^{A \setminus A_j}} \int_{\underline{V} \in \mathscr{S}_E^{A,A_j}} \prod_{y \in A_j} f(V_y)\, d\mu^{(j)}(\underline{V}) \\
&\leqslant \alpha_1^{-1} \sup_{1 \leqslant j \leqslant N} \sup_{\underline{V} \in \mathbb{R}^{A \setminus A_j}} \int_{\underline{V} \in \mathscr{S}_E^{A,A_j}} \prod_{y \in A_j} f(V_y)\, d\mu^{(j)}(\underline{V})
\end{aligned} \tag{7}$$

In the final inequality I have used the assumption (5) on A.

The integral over $\mathscr{S}_E^{A,A_j}$ in (5) must now be estimated. The unit normal to $\mathscr{S}_E^{A,A_j}$ in $\mathbb{R}^{A_j}$ is, by first-order perturbation theory, proportional to $|\psi_{A_j}|^2$. Lemma 1 implies that at all points on $\mathscr{S}_E^{A,A_j} \cap [\operatorname{supp} f]^{A_j}$ at least two components of this unit normal, say n_{x_1} and n_{x_2}, are bounded away from zero by $\beta^2(C, d) > 0$. So write $\mathscr{S}_E^{A,A_j} \cap [\operatorname{supp} f]^{A_j}$ as a disjoint union of at most $\binom{|A_j|}{2}$ surfaces $T_E^{A,A_j,\{x_1,x_2\}}$ in $\mathbb{R}^{A_j}$ on which n_{x_1} and n_{x_2} are bounded away from zero by $\beta^2(C, d)$, i.e.,

$$\mathscr{S}_E^{A,A_j} \cap [\operatorname{supp} f]^{A_j} = \mathring{\bigcup_{x_1 \neq x_2 \in A_j}} T_E^{A,A_j,\{x_1,x_2\}}$$

Integrals over the $T_E^{A,A_j,\{x_1,x_2\}}$ may be bounded by Lemma 1. To see that this is possible, change variables. The intersection of $T_E^{A,A_j,\{x_1,x_2\}}$ with any hyperplane of specified $\{V_w\}_{w\neq x_1,x_2}$ is a subset of $\mathscr{S}_E^{A,\{x_1,x_2\}}$. So, by a change of variables, the integral

$$\int_{\underline{V}\in T_E^{A,A_j,\{x_1,x_2\}}} \prod_{y\in A_j} f(V_j)\, d\mu^{(j)}(\underline{V})$$

may be written as

$$\int \prod_{\substack{w\in A_j\\ w\neq x_1,x_2}} f(V_z)\, dV_z \int_{(V_{x_1},V_{x_2})\in\mathscr{S}_E^{A,\{x_1,x_2\}}} f(V_{x_1})\, f(V_{x_2}) \times J_{x_1,x_2}(V_{x_1}, V_{x_2}; \{V_w\}_{w\neq x_1,x_2})\, dl \tag{8}$$

in which $\mathscr{S}_E^{A,\{x_1,x_2\}}$ is regarded as a $\{V_w\}_{w\neq x_1,x_2}$-dependent curve in $\mathbb{R}^{\{x_1,x_2\}}\simeq\mathbb{R}^2$, and dl is arc length. Here $J_{x_1,x_2}(V_{x_1}, V_{x_2}; \{V_w\}_{w\neq x_1,x_2})$ is a Jacobian factor on $\mathscr{S}_E^{A,\{x_1,x_2\}}$. In terms of the unit normal vector $\underline{n}(\underline{V})$ to $T_E^{A,A_j,\{x_1,x_2\}}\cap[-C, C]^{A_j}$ in $\mathbb{R}^{A_j}$, it equals

$$\begin{cases} |A_j|^{-1}(n_{x_1}^2+n_{x_2}^2)^{-1/2} \sum\limits_{x\in A_j} n_x & \text{if} \quad (V_{x_1}, V_{x_2})\in T_E^{A,A_j,\{x_1,x_2\}} \\ 0 & \text{if} \quad (V_{x_1}, V_{x_2})\notin T_E^{A,A_j,\{x_1,x_2\}} \end{cases} \tag{9}$$

As the unit normal vector $\underline{n}$ has components $n_{x_1}, n_{x_2} > \beta^2(C, d)$, at all points $(V_{x_1}, V_{x_2})\in\mathscr{S}_E^{A,\{x_1,x_2\}}\subset\mathbb{R}^2$,

$$J_{x_1,x_2}(V_{x_1}, V_{x_2}; \{V_w\}_{w\neq x_1,x_2}) \leqslant (2\beta^4)^{-1/2}$$

by (9). Moreover, the two-component unit normal vector $\underline{m}$ to the portion of the curve $\mathscr{S}_E^{A,\{x_1,x_2\}}\subset\mathbb{R}^2$ on which the integrand of (8) is nonzero, i.e., $\mathscr{S}_E^{A,\{x_1,x_2\}}\cap T_E^{A,A_j,\{x_1,x_2\}}$ has its two components $m_{x_1}, m_{x_2} > \beta^2(C, d)$ as well. [In fact, $(m_{x_1}, m_{x_2}) = (n_{x_1}, n_{x_2})/(n_{x_1}^2+n_{x_2}^2)^{1/2}$.] So Lemma 1 applies to the integral (8) and yields

$$\int_{\underline{V}\in T_E^{A,A_j,\{x_1,x_2\}}} \prod_{y\in A_j} f(V_y)\, d\mu^{(j)}(\underline{V}) \leqslant kc(\beta)\, \|f\|_2^2 (2\beta^4)^{-1/2}$$

in which $kc(\beta)$ denotes the bounding constant of Lemma 1. As remarked above, $k=1$ in the finite-difference case, for $\mathscr{S}_E^{A,\{x_1,x_2\}}$ is a hyperbola.

Hence the integral over $\mathscr{S}_E^{A,A_j}$ in (7) is bounded by

$$\binom{|A_j|}{2} kc(\beta)(2\beta^4)^{-1/2}\, \|f\|_2^2$$

And by (7),

$$\mathscr{E}\rho_{\Lambda}(E) \leqslant \alpha_1^{-1} \binom{|\Lambda_0|}{2} kc(\beta)(2\beta^4)^{-1/2} \|f\|_2^2 \tag{10}$$

Therefore the desired $p=2$ bound on $\mathscr{E}\rho_{\Lambda}$ holds uniformly in N, which governs the size of Λ. The $p=2$ case of Theorem 1 is proven. ▮

Proof of Theorem 1′, $p \geqslant 2$ case. The $p \geqslant 2$ case extends the $p=2$ case. It relies on the following two lemmas, which extend Lemmas 1 and 2, respectively.

Lemma 3. Let $\mathscr{S}$ be a differentiable surface in $\mathbb{R}^p$. Suppose that: (i) The unit normal $\underline{m}$ to $\mathscr{S}$ has all p components nonnegative and bounded bounded below by $\beta^2 > 0$. (ii) $\mathscr{S}$ intersects any line parallel to a coordinate axis at most k times.

Then there exists a constant $c = c(\beta)$ such that

$$\int_{\mathscr{S}} \prod_{j=1}^{p} f(V_j)\, g(\underline{V})\, dA \leqslant kc \|f\|_{p'}^{p} \|g\|_{\infty}$$

Remark. By surface is meant locally a finite union of embedded submanifolds of codimension one. Such a definition allows finite self-intersections.

Proof. Deferred to Appendix A.

We also have a $p \geqslant 2$ generalization of Lemma 2, as follows:

Lemma 4. For some $\beta_p(C) > 0$ and some finite set $\Lambda_0'(p) \subset \mathbb{Z}^d$, the following is true. If $\Lambda_0 \subset \Lambda$ is congruent to $\Lambda_0'(p)$ and ψ is an eigenfunction of H_{Λ} with eigenvalue $E \in [-C-2d, C+2d]$, then there exist sites $x_1, \ldots, x_p \in \Lambda_0$ such that $|\psi_{\Lambda_0}(x_k)| \geqslant \beta_p$, $1 \leqslant k \leqslant p$. The quantity ψ_{Λ_0} denotes the normalized (in the l^2 sense) restriction of ψ to Λ_0.

Proof. Deferred to Appendix B. ▮

Given the set $\Lambda_0'(p)$ of Lemma 4, one chooses Λ to be a union of N translates Λ_j of $\Lambda_0'(p)$, such that $|\Lambda| > \alpha_1 N |\Lambda_0'(p)|$. Then

$$\mathscr{E}\rho_{\Lambda}(E) \leqslant \alpha_1^{-1} \sup_{1 \leqslant j \leqslant N} \sup_{\underline{V} \in \mathbb{R}^{\Lambda \setminus \Lambda_j}} \int_{\underline{V} \in \mathscr{S}_E^{\Lambda, \Lambda_j}} \prod_{y \in \Lambda_j} f(V_y)\, d\mu^{(j)}(\underline{V})$$

as in the $p=2$ case.

By first-order perturbation theory, the unit normal vector $n(\cdot)$ to $\mathscr{S}_E^{\Lambda, \Lambda_j}$

in $\mathbb{R}^{\Lambda_j}$ equals $|\psi_{\Lambda_j}(\cdot)|^2$. So by Lemma 4, the integral over $\mathscr{S}_E^{\Lambda,\Lambda_j}$ may be written as a sum of

$$\binom{|\Lambda_0'(p)|}{p}$$

terms, namely

$$\sum_{\substack{A \subset \Lambda_j \\ |A| = p}} \int_{\underline{V} \in T_E^{\Lambda,\Lambda_j,A}} \prod_{y \in \Lambda_j} f(V_y)\, d\mu^{(j)}(V) \tag{11}$$

Here the $T_E^{\Lambda,\Lambda_j,A}$ are defined to be surfaces in $\mathbb{R}^{\Lambda_j}$ on which the p components $\{n_x\}_{x \in A}$ of the normal are bounded below by β_p^2. They provide a disjoint partition of the relevant portion of $\mathscr{S}_E^{\Lambda,\Lambda_j}$, i.e.,

$$\mathscr{S}_E^{\Lambda,\Lambda_j} \cap [\operatorname{supp} f]^{\Lambda_j} = \mathop{\overset{\circ}{\bigcup}}_{\substack{A \subset \Lambda_j \\ |A| = p}} T_E^{\Lambda,\Lambda_j,A}$$

By a change of variables the integrals over $T_E^{\Lambda,\Lambda_j,A}$ may be expressed in terms of integrals over surfaces in $\mathbb{R}^p$, namely the intersections of $T_E^{\Lambda,\Lambda_j,A}$ with hyperplanes of constant $\{V_w\}_{w \in \Lambda \setminus A}$. These are subsets of $\mathscr{S}_E^{\Lambda,A}$. The integral (11) may thus be written as

$$\int \prod_{w \in \Lambda \setminus A} f(V_w)\, dV_w \int_{\underline{V} \in \mathscr{S}_E^{\Lambda,A}} \prod_{x \in A} f(V_x)\, J_A(\underline{V}; \{V_w\}_{w \in \Lambda \setminus A})\, dA \tag{12}$$

in which $\mathscr{S}_E^{\Lambda,A}$ is regarded as a $\{V_w\}_{w \in \Lambda \setminus A}$-dependent surface in $\mathbb{R}^A \simeq \mathbb{R}^p$, and dA is surface area. In the sense of Lemma 3, it has $k = 1$ in the finite-difference case. For it is the solution set in $\mathbb{R}^A$ of the equation $\det(E - H) = 0$, i.e., the zero set of a polynomial linear in each V_x. On the support of J_A, it has exactly one intersection with any line parallel to a coordinate axis in $\mathbb{R}^p$.

Similarly to the $p = 2$ case, the Jacobian factor $J_A(\underline{V}, \{V_w\}_{w \in \Lambda \setminus A})$ equals

$$\begin{cases} |\Lambda_j|^{-1} \left(\sum_{x \in A} n_x^2 \right)^{-1/2} \sum_{x \in \Lambda_j} n_x & \text{if } \quad V \in T_E^{\Lambda,\Lambda_j,A} \\ 0 & \text{if } \quad \underline{V} \notin T_E^{\Lambda,\Lambda_j,A} \end{cases}$$

As $n_x > \beta_p^2$ for all $x \in A$,

$$J_A(\underline{V}; \{V_w\}_{w \in \Lambda \setminus A}) \leqslant (p\beta_p^4)^{-1/2}$$

Moreover, by Lemma 4, the unit normal vector $\underline{m}$ to $\mathscr{S}_E^{A,A}$ in $\mathbb{R}^p$ has its p components $\{m_x\}_{x\in A} \propto \{n_x\}_{x\in A}$ bounded below by $\beta_p^2 > 0$. So Lemma 3 applies to the integral (12) and proves

$$\int_{\underline{V}\in T_E^{A,A_j,A}} \prod_{y\in A_j} f(V_y)\, d\mu^{(j)}(\underline{V}) \leqslant kc(\beta_p)\, \|f\|_{p'}^{p}\, (p\beta_p^4)^{-1/2}$$

Here $c(\beta_p)$ is the p-dependent bounding constant of Lemma 3. Substitution into (11) proves that

$$\mathscr{E}\rho_A(E) \leqslant \alpha_1^{-1} \binom{|A_0'(p)|}{p} kc(\beta)(p\beta_p^4)^{-1/2}\, \|f\|_{p'}^{p} \tag{13}$$

uniformly in N, i.e., in $|A|$. The general case of Theorem 1 is proven. ∎

4. ONE-DIMENSIONAL CONTINUUM RESULTS

The following theorem states bounds on the DOS of one-dimensional continuum random Schrödinger operators which parallel those of Theorem 1.

Theorem 2. Suppose $\operatorname{supp} f \subset [-C, C]$. In the $d=1$ continuum case, for all integer $p \geqslant 1$ there exists a function $c_p(C, E)$, continuous in E, such that $\rho(E) \leqslant c_p(C, E)\, \|f\|_{p'}^{p}$ for all E.

Proof. As in the finite-difference case, it suffices to show that the bounds on ρ hold with ρ replaced by $\mathscr{E}\rho_A$ for A arbitrarily large, i.e., that the following theorem holds.

Theorem 2′. Suppose $\operatorname{supp} f \subset [-C, C]$. In the $d=1$ continuum case, for all integer $p \geqslant 1$ one can choose a sequence of boxes $\{A(N; p) \subset \mathbb{Z}\}_{N=1}^{\infty}$ converging to $\mathbb{Z}$ such that for some continuous $c_p = c_p(C, E)$, $\mathscr{E}\rho_{A(N;p)}(E) \leqslant c_p\, \|f\|_{p'}^{p}$ for all N.

Proof. The proof of Theorem 1′ was so crafted as to serve equally well as a proof of Theorem 2′. The following discussion assumes that the reader has read and understood the proof of Theorem 1′.

The isoeigenvalues surface $\mathscr{S}_E^A$ defined in (1) is now to be interpreted in terms of the continuum Hamiltonian $H_A := -\Delta_{A,D} + V(\cdot)$, which acts on $L^2(\bar{A})$. As a consequence, $\mathscr{S}_E^A$ is no longer an algebraic variety. For example, if $|A| = 2$, then $\mathscr{S}_E^A$ will no longer be a hyperbola.

But essentially all the case-dependent parts of the proof of Theorem 1′ were segregated in Lemma 4. To make the proof of Theorem 1′ serve as a proof of Theorem 2′, Lemma 4 must be replaced by the following:

Lemma 5. For all integer $p \geqslant 1$ there exist (i) $\beta_p(C, E) > 0$ continuous, with no E dependence if $p = 1$, and (ii) a finite set $\Lambda_0'(p) \subset \mathbb{Z}$; such that the following is true. If $\Lambda_0 \subset \Lambda$ is congruent to $\Lambda_0'(p)$ and $\psi \in L^2(\bar{\Lambda})$ is an eigenfunction of H_Λ with eigenvalue $E \geqslant -C$, then there exist sites $x_1, ..., x_p \in \Lambda_0$ such that

$$\int_{\Lambda(x_k)} |\psi_{\Lambda_0}(\cdot)|^2 \geqslant \beta_p^2$$

$1 \leqslant k \leqslant p$. Here ψ_{Λ_0} denotes the normalized (in the L^2 sense) restriction of ψ to $\bar{\Lambda}_0$.

Proof. Deferred to Appendix C, where it is shown that an interval of p sites can serve as $\Lambda_0'(p)$.

By Lemma 3 and 5, the analogue of (13) is

$$\mathscr{E}\rho_\Lambda(E) \leqslant \alpha_1^{-1} \binom{|\Lambda_0'(p)|}{p} kc(\beta_p(C, E))(p\beta_p^4)^{-1/2} |\Lambda_0'(p)| \, \|f\|_{p'}^p \tag{14}$$

It remains only to choose k, a bound on the number of intersections of any line parallel to a coordinate axis in $\mathbb{R}^A \simeq \mathbb{R}^p$ with the surface $\mathscr{S}_E^{A,\Lambda}$. Of course $\mathscr{S}_E^{A,\Lambda}$ is defined as in Section 2, with A the p-element subset of Λ guaranteed to exist by Lemma 5.

In the continuum case, there is no polynomial in the $\{V_x\}_{x \in A}$ of which $\mathscr{S}_E^{A,\Lambda}$ is the zero set, and we cannot set $k = 1$. But as $d = 1$, a tight bound on k can be found by explicit calculation.

Definition. If $A \subset \Lambda \subset \mathbb{Z}^d$ and $E \in \mathbb{R}$, define $-\Delta_{A,\Lambda;E}$ as the Friedrichs extension of the symmetric operator $-\Delta$ on a domain $\mathscr{D}_E \subset L^2(\bar{A})$, namely,

$\mathscr{D}_E := \{f \mid f = g \restriction \bar{A}$ for some $g \in C^1(\bar{\Lambda})$ which
(i) is C^2 on the interiors of the cells $\{\Lambda(x)\}_{x \in \Lambda}$, with
continuous extensions of its second derivatives to $\partial\Lambda(x)$;
(ii) obeys Dirichlet boundary conditions on $\bar{\Lambda}$;
(iii) satisfies $(-\Delta + V - E)\, g = 0$ on $\bar{\Lambda} \backslash \bar{A}\}$

Definition. $H_{A,\Lambda;E} := -\Delta_{A,\Lambda;E} + \sum_{y \in A} V_y \chi_{\Lambda(y)}$. The $H_{A,\Lambda;E}$ is a self-adjoint operator on $L^2(\bar{A})$.

Definition. By $\mathscr{S}_E^{A,\Lambda;E}$ one denotes $\{\underline{V} \in \mathbb{R}^A \mid E \in \sigma(H_{A,\Lambda;E})\}$. It may be regarded as a $\{V_x\}_{x \in \Lambda \backslash A}$-dependent surface in $\mathbb{R}^A \simeq \mathbb{R}^p$, like $\mathscr{S}_E^{A,\Lambda}$.

Proposition 2. $\mathscr{S}_E^{\Lambda,A;E} \subset \mathscr{S}_E^{\Lambda,A}$, and $\underline{V} \in \mathscr{S}_E^{A} \backslash \mathscr{S}_E^{\Lambda,A;E}$ only if $N_x((\underline{V}, \{V_x\}_{x\in\Lambda\setminus A}))=0$ for all $x\in A$. The $\underline{N}((\underline{V}, \{V_x\}_{x\in\Lambda\setminus A}))$ denotes the unit normal vector to $\mathscr{S}_E^{A} \subset \mathbb{R}^A$ at the point $(\underline{V}, \{V_x\}_{x\in\Lambda\setminus A})$.

Proof. Recall that $\mathscr{S}_E^{\Lambda,A} := \{\underline{V}\in\mathbb{R}^A \mid E\in\sigma(H_\Lambda)\}$. So in effect the proposition states that if $(\underline{V}, \{V_x\}_{x\in\Lambda\setminus A})\in\mathscr{S}_E^{\Lambda}$ and $N_x((\underline{V}, \{V_x\}_{x\in\Lambda\setminus A}))\neq 0$ for at least one $x\in A$, then $H_{\Lambda,A;E}$ will have E as an eigenvalue.

If $(\underline{V}, \{V_x\}_{x\in\Lambda\setminus A})\in\mathscr{S}_E^{\Lambda}$, then H_Λ will have an eigenfunction $\psi_E\in L^2(\bar{\Lambda})$ of eigenvalue E. By (2.5), as $N_x((\underline{V}, \{V_x\}_{x\in\Lambda\setminus A}))\neq 0$ for at least one $x\in A$, its restriction $\psi_E \restriction \bar{A}$ will have positive norm in $L^2(\bar{A})$. Moreover, it will lie in $\mathscr{D}_E$. So $\psi_E \restriction \bar{A}$ is in the domain of $-\Delta_{\Lambda,A;E}$ and satisfies $H_{\Lambda,A;E}\psi_E = E\psi_E$. ∎

As far as Eq. (11) and its successors are concerned, Proposition 2 implies that $\mathscr{S}_E^{\Lambda,A}$ can be replaced by $\mathscr{S}_E^{\Lambda,A;E}$. For, by the definition (5.5) of surface measure on $\mathscr{S}_E^{\Lambda,A}$, $\mathscr{S}_E^{\Lambda,A;E} \backslash \mathscr{S}_E^{\Lambda,A}$ is assigned measure zero.

This replacement of $\mathscr{S}_E^{\Lambda,A}$ by $\mathscr{S}_E^{\Lambda,A;E}$ aids in choosing the bound k. If $|\cdot|$ denotes cardinality, we may choose

$$k \geqslant \max_{x\in A} \sup_{V_y,\, y\neq x} |\{V_x\in[-C, C] \mid \underline{V}\in\mathscr{S}_E^{\Lambda,A;E}\}|$$
$$= \max_{x\in A} \sup_{V_y,\, y\neq x} \left|\left\{V_x\in[-C, C] \,\middle|\, E\in\sigma(-\Delta_{\Lambda,A;E}) + \sum_{y\in A} V_y\chi_{A(y)}\right\}\right| \tag{15}$$

If $C_1 = C+\varepsilon$ for $\varepsilon>0$, define modified potentials $\underline{V}_\pm$ by

$$(V_\pm)_x = \begin{cases} V_y, & y\neq x \\ \pm C_1, & y=x \end{cases}$$

Then, by standard arguments[4]

$$|\{V_x\in[-C, C] \mid \underline{V}\in\mathscr{S}_E^{\Lambda,A;E}\}|$$
$$\leqslant N(E; -\Delta_{\Lambda,A;E} + \underline{V}_-) - N(E; -\Delta_{\Lambda,A;E} + \underline{V}_+)$$

In this equation $N(E; T)$ signifies the number of eigenvalues of a self-adjoint operator T less than or equal to E. Moreover, as $\underline{V}_- > -C_1$ and $\underline{V}_+ < C_1$,

$$N(E; -\Delta_{\Lambda,A;E} + \underline{V}_-) \leqslant N(E; -\Delta_{\Lambda,A;E} - C_1)$$
$$N(E; -\Delta_{\Lambda,A;E} - \underline{V}_+) \geqslant N(E; -\Delta_{\Lambda,A;E} + C_1)$$

So

$$|\{V_x \in [-C, C] \mid \underline{V} \in \mathscr{S}_E^{A,A;E}\}| \leqslant N(E; -\Delta_{A,A;E} - C_1) - N(E; -\Delta_{A,A;E} + C_1)$$

or equivalently

$$|\{V_x \in [-C, C] \mid \underline{V} \in \mathscr{S}_E^{A,A;E}\}| \leqslant N(E + C_1; -\Delta_{A,A;E}) - N(E - C_1; -\Delta_{A,A;E}) \tag{16}$$

So it suffices to bound the number of eigenvalues of the operator $-\Delta_{A,A;E}$ contained in the interval $[E - C_1, E + C_1]$. This bound must be uniform in the choice of $\{V_x\}_{x \in \Lambda \setminus A}$.

In the one-dimensional case such a bound may be found by explicit solution of the eigenvalue equation

$$(-\Delta_{A,A;E} - \lambda)\psi = 0 \tag{17}$$

in the domain $\mathscr{D}_E$, where $\mathscr{D}_E$ is specified by $\{V_x\}_{x \in \Lambda \setminus A}$, or equivalently (if $\bar{A} = [a, b]$) by homogeneous linear boundary conditions

$$\begin{aligned} \alpha_1 \psi(a) + \alpha_2 \psi'(a) &= 0 \\ \beta_1 \psi(b) + \beta_2 \psi'(b) &= 0 \end{aligned} \tag{18}$$

for some α_1, α_2, β_1, β_2 depending on $\{V_x\}_{x \in \Lambda \setminus A}$. The solution

$$\psi(x) = A \sin \sqrt{\lambda}\, x + B \cos \sqrt{\lambda}\, x$$

to (17) can be made to satisfy the boundary conditions (18) only if

$$\tan(b - a)\sqrt{\lambda} + \frac{(\alpha_2 \beta_1 + \alpha_1 \beta_2)\sqrt{\lambda}}{\alpha_1 \beta_1 - \alpha_2 \beta_2 \lambda} = 0 \tag{19}$$

The number of solutions $\lambda \in \mathbb{R}$ of (18) within any compact interval is bounded uniformly in α_1, α_2, β_2, β_2, namely in $\{V_x\}_{x \in \Lambda \setminus A}$. In particular, the number of eigenvalues in $[E, -C_1, E + C_1]$ is so bounded by some continuous $k(E)$.

If this E-dependent k is substituted into (16), the majorization of Theorem 2′ results. The bounding constant $c_p(C, E)$ will equal

$$\alpha_1^{-1} \binom{|\Lambda_0'(p)|}{p} k(E)\, c(\beta_p(C, E))(p\beta_p^4)^{-1/2}\, |\Lambda_0'(p)| \quad \blacksquare$$

Note that our continuum restriction to $d=1$ arose only from the need for the bound (15) on k to be finite. This finiteness follows by explicit computation only if $d=1$.

APPENDIX A. THE PROOF OF LEMMA 3

If $1 \leqslant k,\ l \leqslant p$, write $\hat{V}_{kl} = (V_1,\ldots, \hat{V}_k,\ldots, \hat{V}_l,\ldots, V_p) \in \mathbb{R}^{p-2}$. By changing variables, the surface integral

$$\int_S \prod_{j=1}^{p} f_j(V_j)\, g(\underline{V})\, dA$$

$$= \int_{\hat{V}_{kl} \in \mathbb{R}^{p-2}} \prod_{\substack{j=1 \\ j \neq k,l}} f_j(V_j) \int_{(V_k, V_l) \in S_{kl}} (m_k^2 + m_l^2)^{-1/2}\, g(\underline{V})\, f(V_k)\, f(V_l)\, dl \quad \text{(A1)}$$

in which $S_{kl} := \{(V_k, V_l) \in \mathbb{R}^2 \mid \underline{V} \in S\}$ is a plane curve in $\mathbb{R}^2$, dl is length measure, and the factor $(m_k^2 + m_l^2)^{-1/2}$ arises as a Jacobian. By assumption $(m_k^2 + m_l^2)^{-1/2} \leqslant (2\beta^4)^{-1/2}$.

Moreover, S_{kl} intersects any line parallel to either coordinate axis in $\mathbb{R}^2$ no more than k times. So Lemma 1 applied to (A1) proves

$$\int_S \prod_{j=1}^{p} f_j(V_j)\, g(\underline{V})\, dA \leqslant (2\beta^4)^{-1/2} \|g\|_\infty \|f_k\|_2 \|f_l\|_2 \prod_{\substack{j=1 \\ j \neq k,l}}^{p} \|f_j\|_1 \quad \text{(A2)}$$

The choice of k and l in (A2) was arbitrary, so $\binom{p}{2}$ bounds result. One may interpolate among them; the following $p=3$ case will serve to illustrate the process. Define a multilinear map $T\colon \times^3 L^1 \cap L^2(\mathbb{R}) \to \mathbb{C}$ by

$$T(f_1, f_2, f_3) := \int_S \prod_{j=1}^{3} f_j(V_j)\, g(\underline{V})\, dA$$

Then (A2) states that

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|g\|_\infty \|f_1\|_1 \|f_2\|_2 \|f_3\|_2 \quad \text{(A3)}$$

and

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|g\|_\infty \|f_1\|_2 \|f_2\|_1 \|f_3\|_2 \quad \text{(A4)}$$

For f_2, f_3 fixed, consider the adjoint map:

$$T'\colon \mathbb{C} \times L^1 \cap L^2(\mathbb{R}) \times L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^\infty \cap L^2(\mathbb{R})$$

It must satisfy

$$\|T'(z, f_2, f_3)\|_\infty \leqslant (2\beta^4)^{-1/2} |z| \, \|f_2\|_2 \, \|f_3\|_2 \, \|g\|_\infty$$
$$\|T'(z, f_2, f_3)\|_2 \leqslant (2\beta^4)^{-1/2} |z| \, \|f_2\|_1 \, \|f_3\|_2 \, \|g\|_\infty$$

The Riesz–Thorin Theorem[4] implies

$$\|T'(z, f_2, f_3)\|_4 \leqslant (2\beta^4)^{-1/2} |z| \, \|f_2\|_{4/3} \, \|f_3\|_2 \, \|g\|_\infty$$

which implies

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|f_1\|_{4/3} \, \|f_2\|_{4/3} \, \|f_3\|_2 \, \|g\|_\infty \tag{A5}$$

Similarly,

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|f_1\|_{4/3} \, \|f_2\|_2 \, \|f_3\|_{4/3} \, \|g\|_\infty \tag{A6}$$

Just as we interpolated between (A3) and (A4) to get (A5), so can we interpolate between (A5) and (A6) to obtain

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|f_1\|_{4/3} \, \|f_2\|_{8/5} \, \|f_3\|_{8/5} \, \|g\|_\infty$$

An obvious limit argument involving further interpolations proves

$$|T(f_1, f_2, f_3)| \leqslant (2\beta^4)^{-1/2} \|f_2\|_{3/2} \, \|f_2\|_{3/2} \, \|f_3\|_{3/2} \, \|g\|_\infty \tag{A7}$$

Choosing $f_1 = f_2 = f_3 = f$ in (A7) yields

$$|T(f, f, f)| \leqslant (2\beta^4)^{-1/2} \|f\|_{3/2}^3 \, \|g\|_\infty$$

as desired. For $p > 3$ one can similarly deduce

$$|T(f_1, \ldots, f_p)| \leqslant (2\beta^4)^{-1/2} \|g\|_\infty \prod_{j=1}^{p} \|f_j\|_{p'}$$

so that Lemma 3 follows. ∎

APPENDIX B. THE PROOF OF LEMMA 4

In effect, one must prove that if $\psi \in l^2(\Lambda)$ is an eigenfunction of H_Λ with eigenvalue E, there must exist p elements of $\Lambda_0 \subset \Lambda$ at which $|\psi|$ is large, "large" having a meaning uniform in $\{V_y\}_{y \in \Lambda}$. The shape of Λ_0, i.e., $\Lambda_0'(p)$, is to be chosen to ensure this.

Finding such a $\Lambda_0'(p)$ is not a trivial matter, as a simple example should make clear. Take $d = 2$ and Λ_0 a square in $\mathbb{Z}^2$, e.g., $[1, L]^2$. Even in the case $V=0$ (so that $H=-\Delta$) such a Λ_0 fails to satisfy the conditions of Lemma 2 for any $p\geqslant 2$. To see this, suppose $\psi_{\Lambda_0}(x)$ were defined on Λ_0 to be

$$\begin{cases} 1/\sqrt{2}, & x=(1,2) \\ -1/\sqrt{2}, & x=(2,1) \\ 0, & \text{otherwise} \end{cases}$$

If Λ were a square centered on and properly containing Λ_0, ψ_{Λ_0} could easily be extended to ψ_Λ, an eigenfunction of H_Λ with zero eigenvalue. But $\psi_\Lambda(x)$ would be nonzero at only two values of $x\in\Lambda_0$.

The familiar polyhedra in $\mathbb{Z}^d$ are unsuitable candidates for $\Lambda_0'(p)$ because their boundaries are too jagged. The following adjacency condition on Λ_0' will prove sufficient for our needs, however.

Definition. A set $\mathscr{S}\subset\mathbb{Z}^d$ is said to have property $P(q,d)$ if for every $T\subset\mathscr{S}$, $|T|\leqslant q$, there exists $z\in\mathscr{S}\backslash\partial\mathscr{S}\backslash T$ such that z is adjacent to one and only one element of T.

Remark. $x, y\in\mathbb{Z}^d$ are said to be adjacent if $|x-y|=1$. (This definition of adjacency is ultimately due to the choice of $-\Delta$ as unperturbed Hamiltonian.) $\partial\mathscr{S}$ is defined to be the set of elements of $\mathscr{S}$ adjacent to $\mathbb{Z}^d\backslash\mathscr{S}$.

Lemma B1. If a finite set $\Lambda_0'(p)\subset\mathbb{Z}^d$ has property $P(p-1,d)$, it will satisfy the requirements of Lemma 4.

Proof. If Λ_0 is congruent to $\Lambda_0'(p)$, consider the restricted eigenfunction ψ_{Λ_0} of Lemma 4. By definition it satisfies

$$-\sum_{\substack{y\in\Lambda_0\\|x-y|=1}}\psi_{\Lambda_0}(y)+[V_x-E]\,\psi_{\Lambda_0}(x)=0$$

if $x\notin\partial\Lambda_0$. We have $E\in[-C-2d, C+2d]$, $V_x-E\in[-2C-2d, 2C+2d]$ for all $x\in\Lambda_0$.

Let $x_1\in\Lambda_0$ be the element at which $|\psi_{\Lambda_0}|$ takes its maximum value. (If there is more than one, choose one arbitrarily.) Note that $|\psi_{\Lambda_0}(x_1)|\geqslant|\Lambda_0'(p)|^{-1/2}$. By assumption there exists $y_2\in\Lambda_0\backslash\partial\Lambda_0$ adjacent to x_1, as follows it $T=T_1:=\{x_1\}$. The triangle inequality applied to (10) implies that either:

(i) $|\psi_{\Lambda_0}(y_2)|\geqslant(2d)^{-1}(2C+2d)^{-1}\,|\psi_{\Lambda_0}(x_1)|$; or

(ii) $|\psi_{\Lambda_0}(y_2')|\geqslant(2d)^{-1}\,|\psi_{\Lambda_0}(x_1)|$ for some $y_2'\neq x_1$ adjacent to y_2.

If (i), let $x_2 := y_2$. If (ii), let $x_2 := y_2'$. In either case

$$|\psi_{\Lambda_0}(x_2)| \geqslant (2d)^{-1}(2C+2d)^{-1}\, |\psi_{\Lambda_0}(x_1)| \geqslant (2d)^{-1}(2C+2d)^{-1}\, |\Lambda_0'(p)|^{-1/2}$$

By induction one may generate an ordered set $T_p = \{x_1,..., x_p\}$ of elements of Λ_0 such that

$$|\psi_{\Lambda_0}(x_{k+1})| \geqslant (2d)^{-1}(2C+2d)^{-1}\, |\psi_{\Lambda_0}(x_k)|, \qquad 1 \leqslant k \leqslant p-1$$

If

$$\beta_p := (2d)^{-p}(2C+2d)^{-p}\, |\Lambda_0'(p)|^{-1/2}$$

then Lemma B1 is proven. ∎

By Lemma B1, to prove Lemma 3, it suffices to construct a finite set $\Lambda_0'(p) \subset \mathbb{Z}^d$ with the property $P(p-1, d)$. The inductive construction I shall use is simple, but yields a $\Lambda_0'(p)$ rather larger than necessary.

Definition. If $\mathscr{S} \subset \mathbb{Z}^d$, define $o(\mathscr{S}) := \mathscr{S} \backslash \partial \mathscr{S}$. One writes $o^2(\mathscr{S})$ for $(\mathscr{S}\backslash\partial\mathscr{S})\backslash\partial(\mathscr{S}\backslash\partial\mathscr{S})$, etc. By $o^0(\mathscr{S})$ is meant $\mathscr{S}$ itself.

Definition. If $\mathscr{S} \subset \mathbb{Z}^d$, define $E_{q,k}(\mathscr{S}) \subset \mathbb{Z}^{d+1}$ to be the set

$$\{(\underline{x}, y) \in \mathbb{Z}^d x \mathbb{Z} \mid \underline{x} \in \mathscr{S} \text{ and } |y| \leqslant q-1, \text{ or } \underline{x} \in o^{|y|-q+1}(\mathscr{S}) \text{ and } q \leqslant |y| \leqslant q+k-1\}$$

Remark 1. $E_{q,k}(\mathscr{S})$ may be thought of as a cylinder with base $\mathscr{S}$ and height $2q-1$, capped on either end by truncated cones of height at most k. The top surfaces of these truncated cones are $o^k(\mathscr{S})$, unless $o^k(\mathscr{S})$ is empty.

Remark 2. $o^r \circ E_{q,k} = E_{q,k-r} \circ o^r$ for all $r \leqslant k$.

Lemma B2. Let $\mathscr{S} \subset \mathbb{Z}^d$. Suppose that $o^r(\mathscr{S})$ has the property $P(q, d)$ for all $0 \leqslant r \leqslant 2q-2$ and that $o^{2q-1}(\mathscr{S})$ is nonempty. Then for all $k \geqslant 2q-1$, $E_{q,k}(\mathscr{S}) \subset \mathbb{Z}^{d+1}$ has the property $P(q, d+1)$.

This lemma may be used inductively to construct, for all d, finite sets $\Lambda_0'(p) \subset \mathbb{Z}^d$ with the desired property $P(p-1, d)$. The following will serve to illustrate the process; indeed, to begin the induction on d.

Denote by I_l an interval of length l in $\mathbb{Z}$. By examination, it has the property $P(q, 1)$ for all $q \leqslant l/2-1$, i.e., $l \geqslant 2q+2$. As $o^r(I_l) = I_{l-2r}$, $o^r(I_l)$ has the property $P(q, 1)$ for all $r \leqslant l/2-q-1$. Hence, by Lemma B2, if $l/2-q-1 \geqslant 2q-2$, i.e., $l \geqslant 6q-2$, then $E_{q,k}(I_l) \subset \mathbb{Z}^2$ will have property $P(q, 2)$ for all $k \geqslant 2q-1$.

Moreover, as

$$o^s E_{q,k}(I_l) = E_{q,k-s}(o^s I_l) = E_{q,k-s} I_{l-2s}$$

then $o^s(E_{q-k}, I_l) \subset \mathbb{Z}^2$ will have property $P(q, 2)$ if

$$l - 2s \geq 6q - 2, \quad \text{i.e.,} \quad s \leq l/2 - 3q + 1$$
$$k - s \geq 2q - 1, \quad \text{i.e.,} \quad k \geq s + 2q - 1$$

In particular, $o^s(E_{q-k}, I_l)$ will have property $P(q, 2)$ for all $s \leq 2q - 2$ if

$$2q - 2 \leq l/2 - 3q + 1, \quad \text{i.e.,} \quad l \geq 10q - 6$$
$$k \geq (2q - 2) + 2q - 1, \quad \text{i.e.,} \quad k \geq 4q - 3$$

So, by Lemma B2, $E_{q,k_2} E_{q,k_1} I_l \subset \mathbb{Z}^3$ will have property $P(q, 3)$ if

$$l \geq 10q - 6, \qquad k_1 \geq 4q - 3, \qquad k_2 \geq 2q - 1$$

And so forth.

In general, by induction on d, one has that

$$E_{q,k_{d-1}} E_{q,k_{d-2}} \cdots E_{q,k_2} E_{q,k_1} I_l \subset \mathbb{Z}^d$$

has property $P(q, d)$ if

$$l \geq 4d(q-1) - 2q + 6$$
$$k_j \geq 1 + 2(q-1)(d-j), \qquad 1 \leq j \leq d-1$$

One may choose $\Lambda_0'(p) \subset \mathbb{Z}^d$ to be such a set $E_{q,k_{d-1}} \cdots E_{q,k_1} I_l$, if $q = p - 1$. By Lemma B1, it will satisfy the conditions of Lemma 3. ∎

It only remains to prove Lemma B2, which made possible the inductive step in the construction of $\Lambda_0'(p)$.

Proof of Lemma B2. I shall show that for any q-element subset $T \subset E_{q,k}(\mathscr{S})$ there exists $z \in E_{q,k}(\mathscr{S}) \backslash \partial E_{q,k}(\mathscr{S}) \backslash T$, i.e., $z \in o(E_{q,k}) \backslash T$, such that z is adjacent to one and only one element of T. So $E_{q,k}(\mathscr{S})$ will have property $P(q, d+1)$.

By the definition of adjacency, one may without loss of generality assume that T is "next-nearest-neighbor connected," i.e., that one does not have $T = T_1 \cup T_2$ and $|x_1 - x_2| > \sqrt{2}$ for all $x_1 \in T_1$, $x_2 \in T_2$. This assumption has strong consequences. The "cylinder" of $E_{q,k}(\mathscr{S}) \subset \mathbb{Z}^{d+1}$ (i.e., $\mathscr{S} \times [-q+1, q-1]$) has vertical height $2q - 1$, and each "cap," e.g.,

$$\{(\underline{x}, y) \mid \underline{x} \in o^{y-q+1}(\mathscr{S}),\ y \in [q, q-1+k]\}$$

by assumption has vertical height $\geq 2q - 1$. As $|T| = q$ and T is next-nearest-neighbor connected, one has immediately that:

(i) T cannot include sites from both caps.

(ii) If T contains sites from the cylinder, it cannot include sites from the portion of a cap with $|y| \geqslant 3q-2$. In particular, it will contain none from the top surface of the cap, the magnitude of the y coordinate of which is at least $3q-2$.

For T cannot extend more than $2q-1$ units in the vertical direction. Hence, one or the other of the following possibilities must be true:

I. T is a subset of a single cap.

II. T is a subset of the union of the cylinder and a single cap, but includes no sites from the top surface of the cap, or indeed any with $|y| \geqslant 3q-2$.

I shall treat these two cases in turn. In both cases, write $T = \{(\underline{x}_l, y_l) \mid 1 \leqslant l \leqslant q\}$.

Case I. Suppose that T is a subset of the upper cap, i.e., of

$$\{(\underline{x}, y) \mid \underline{x} \in o^{y-q+1}(\mathscr{S}),\ y \in [q, q+k-1]\}$$

The case of the lower cap is similar. Let $(\underline{x}_l, y_l)$ be the element of T with the least value of y; if more than one element has this value for its y coordinate, choose one arbitrarily. Define $z := (\underline{x}_l, y_l - 1) \in E_{q,k}(\mathscr{S})$. Then $z \in o(E_{q,k}(\mathscr{S})) \backslash T$, and z is adjacent to one and only one element of T, namely $(\underline{x}_l, y_l)$. These are the desiderata of property $P(q, d+1)$.

Case II. Suppose that T is a subset of the union of the upper cap and the cylinder; the case of the lower cap and the cylinder is similar. Then for all $1 \leqslant l \leqslant q$, y_l lies between $-q+1$ and $3q-3$. Denote by $(\underline{x}_m, y_m)$ the element of T with the largest value of y, and define $T' := \{\underline{x}_l \mid \underline{x}_l \in o^{y_m - q+1}(\mathscr{S})\}$. The set T' is nonempty, for it contains $\underline{x}_m$.

As remarked, $y_m \leqslant 3q-3$. Hence $y_m - q + 1 \leqslant 2q-2$, so $T' \subset o^r(\mathscr{S})$ for some $r \leqslant 2q-2$. By assumption, there exists $\underline{z} \in o^r(\mathscr{S}) \backslash \partial o^r(\mathscr{S})$ such that $\underline{z}$ is adjacent to exactly one element of T', say $\underline{x}_t$. Therefore $(\underline{x}, y_t) \in E_{q,k}(\mathscr{S}) \backslash T$ is adjacent to exactly one element of T, namely $(\underline{x}_t, y_t)$. Moreover, as $\underline{z} \notin \partial o^r(\mathscr{S})$ and $y_t \leqslant 3q-3$, $(\underline{z}, y_t) \notin \partial E_{q,k}(\mathscr{S})$. So $(\underline{z}, y_t) \in E_{q,k}(\mathscr{S}) \backslash \partial E_{q,k}(\mathscr{S}) \backslash T$ and $(\underline{z}, y_t)$ will serve as the desired element $z \in E_{q,k}(\mathscr{S})$. ∎

APPENDIX C. THE PROOF OF LEMMA 5

The $p=1$ case of Lemma 5 is trivially true. One chooses $\Lambda'_0(1) \subset \mathbb{Z}$ to be a singleton, and $\beta_p = 1$.

In general, elementary integral calculus suffices to prove that an interval of p sites will serve as $\Lambda_0'(p)$. The following argument is simple, but yields a minorizing constant β_p far smaller than optimal.

If ψ is an eigenfunction of H_Λ with eigenvalue E, it satisfies the differential equation $-\psi'' + [V(x) - E]\psi = 0$. By integration by parts, for all $x \in \bar{\Lambda}$ such that $[x-\alpha, x+\alpha] \subset \bar{\Lambda}$

$$\psi(x) - \tfrac{1}{2}\psi(x-\alpha) - \tfrac{1}{2}\psi(x+\alpha)$$
$$= -\tfrac{1}{2}\int_{y=x-\alpha}^{x}\int_{z=x}^{y} [V(z)-E]\,\psi(z) - \tfrac{1}{2}\int_{y=x}^{x+\alpha}\int_{z=x}^{y} [V(z)-E]\,\psi(z) \tag{C1}$$

As $|V(y) - E| \leqslant |E| + C$, the Cauchy–Schwartz inequality implies that the right-hand side is bounded by $\alpha^{1/2}[C + |E|]\, \|\psi\|_{2,[x-\alpha,x+\alpha]}$, if $\|\cdot\|_2$ denotes L^2 norm on an interval.

Let I_1, I_2, $I_3 \subset \Lambda_0$ be successive intervals of length α. Then this bound implies

$$|\psi(x) - \tfrac{1}{2}\psi(x-\alpha) - \tfrac{1}{2}\psi(x+\alpha)| \leqslant \alpha^{1/2}[C + |E|]\, \|\psi\|_{2,[x-\alpha,x+\alpha]} \tag{C2}$$

for all $x \in I_2$. Taking the supremum of (C2) over $x \in I_2$ yields

$$\begin{aligned} &\|\psi \restriction I_2 - \tfrac{1}{2}\psi \restriction I_1 - \tfrac{1}{2}\psi \restriction I_2\|_\infty \\ &\quad \leqslant \alpha^{1/2}[C + |E|][\|\psi \restriction I_1\|_2 + \|\psi \restriction I_2\|_2 + \|\psi \restriction I_3\|_2] \end{aligned} \tag{C3}$$

in which the L^p norms are understood to apply to functions defined on intervals of length α. But as such L^p norms satisfy $\|\cdot\|_2 \leqslant \alpha^{1/2}\, \|\cdot\|_\infty$, (C3) implies that

$$\begin{aligned} &\|\psi \restriction I_2 - \tfrac{1}{2}\psi \restriction I_1 - \tfrac{1}{2}\psi \restriction I_3\|_2 \\ &\quad \leqslant \alpha[C + |E|][\|\psi \restriction I_1\|_2 + \|\psi \restriction I_2\|_2 + \|\psi \restriction I_3\|_2] \\ &\quad \leqslant 6\alpha[C + |E|]\max\{\|\psi \restriction I_2\|_2, \|\tfrac{1}{2}\psi \restriction I_1\|_2, \|\tfrac{1}{2}\psi \restriction I_3\|_2\} \end{aligned} \tag{C4}$$

Lemma C1. If I_1, I_2, $I_3 \subset \bar{\Lambda}$ are adjacent intervals of length $\alpha \leqslant 18^{-1}[C + |E|]^{-1}$, then

$$\max\{\|\psi \restriction I_2\|_2, \|\psi \restriction I_3\|_2\} \geqslant 6^{-1}\, \|\psi \restriction I_1\|_2$$
$$\max\{\|\psi \restriction I_2\|_2, \|\psi \restriction I_1\|_2\} \geqslant 6^{-1}\, \|\psi \restriction I_3\|_2$$

Proof. By the triangle inequality applied to (C4). ∎

Lemma 5 now follows from Lemma C1 by induction on p, if one decomposes the intervals $\{\Lambda(x)\}_{x\in\Lambda}$ of $\bar{\Lambda}$ into subintervals of length $\alpha \leqslant 18^{-1}[C+|E|]^{-1}$. One finds $\beta_p \sim 6^{-p/2\alpha} \sim 6^{-9p[C+|E|]}$. ∎

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